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Some special studies on dynamic response of a simply supported beam to impact loads

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(Plate—2)

In this paper broad aspects of impact between a uniform beam supported at its ends and a transversely impinging load at its midspan have been analysed. The effect of the striking velocity of the load on the duration of impact in the light of Hertz's (1927) theory is discussed. The range of application of Cox's (1856) theory with respect to "mass ratio" has been explained. The vibration pattern of the beam, the duration of impact and the energy absorbed by the beam during impact are studied in details both theoretically and experimentally. Photographic method of measurement has been used in the experiments. Experimental results are in excellent agreement with the theory.

Introduction

Investigations on the beam-impact problem have been made by Cox (1856), Timoshenko (1956), Hoppman (1948), Banerjee (1966*a*) and others. Recently the author & Mishra (1968) following the deductions of Banerjee (1966a) has developed a general theory on transverse impact on a simply supported beam. In the present investigation the author using various beam-load combinations has made some special studies in details concerning the displacement, period of vibration, the energy of the beam and the duration of impact for a simply supported beam under central impact. Further, the results of the present analysis of the author have been compared with those of Cox's (1856) theory. Theoretical results have been verified experimentally employing photographic method of measurement

Theory

Let us consider a uniform beam of length l, mass M, Young's modulus E_1, and moment of inertia about the neutral axis I, simply supported at its ends $x = 0$ and $x = l$ and transversely struck at mid-span ($x = a = l/2$) by a load of mass m with a striking velocity V_0

Displacement of the beam and pressure of impact of the load.

The analysis of Das (1968) yields the following expressions given by equations. (1) to (4).

$$\frac{2 \sinh\gamma_s \, . \sin \gamma_s}{\sinh \gamma_s \, . \sin^2 \frac{\gamma_s}{2} - \sin \gamma_s \, . \sinh^2 \frac{\gamma_s}{2}} = \frac{\frac{m}{M} \, . \, \gamma_s}{1 - \frac{E_1}{E_2} \, . \, \frac{m}{M} \, . \, \frac{\gamma_s^4}{l^3}} \qquad \ldots \quad (1)$$

$$q_s = \gamma_s^2 \left(\frac{E_1 I}{M l^3}\right)^{\frac{1}{2}} \qquad \ldots \quad (2)$$

$$y_a = 4V_0 \Sigma \frac{A_s}{q_s} \,.\, \sin q_s t \qquad \ldots \quad (3.1)$$

$$y_1 = 4V_0 \sum \frac{\left(\sinh \gamma_s \,.\, \sin \frac{\gamma_s}{2} \,.\, \sin \gamma_s \frac{x}{l} - \sin \gamma_s \,\sinh \frac{\gamma_s}{2} \,\sinh \gamma_s \frac{x}{l}\right)}{\sinh \gamma_s \,.\, \sin^2 \frac{\gamma_s}{2} - \sin \gamma_s \,.\, \sinh^2 \frac{\gamma_s}{2}} \times \frac{A_s}{q_s} \sin q_s t$$

$$\ldots \quad (3.2)$$

$$y_2 = 4V_0 \sum \frac{\left[\sinh \gamma_s \,.\, \sin \frac{\gamma_s}{2} \,.\, \sin \gamma_s \left(1 - \frac{x}{l}\right) - \sin \gamma_s \,.\, \sinh \frac{\gamma_s}{2} \,.\, \sinh \gamma_s \left(1 - \frac{x}{l}\right)\right]}{\sinh \gamma_s \,.\, \sin^2 \frac{\gamma_s}{2} - \sin \gamma_s \,.\, \sinh^2 \frac{\gamma_s}{2}}$$

$$\times \frac{A_s}{q_s} \,.\, \sin q_s t \qquad \ldots \quad (3.3)$$

$$\frac{1}{A_s} = 1 + 2\frac{M}{m} + \frac{\gamma_s}{\coth \frac{\gamma_s}{2} - \cot \frac{\gamma_s}{2}} \qquad \text{(for hard load)} \qquad \ldots \quad (3.4)$$

$$P = -4mV_0 \Sigma B_s \; q_s \,.\, \sin q_s t \qquad \ldots \quad (4.1)$$

$$\frac{1}{B_s} = 1 + \frac{3mq_s^2}{E_2} + \frac{m\gamma_s^2}{4M}\left[\frac{1}{\cos \gamma_s + 1} - \frac{1}{\cosh \gamma_s + 1}\right] \qquad \ldots \quad (4.2)$$

where, γ_s = pure number (for $s = 1, 2, 3, \ldots$ etc.) representing different modes of vibration, q_s = circular frequency of the vibrating beam, y_a = displacement of the beam at the struck point ($x = a = l/2$), y_1 = displacement of the beam at any point $x < l/2$, y_2 = displacement of the beam at any point $x > l/2$, P = pressure of impact (impact force) of the load, t = variable time, E_2 = shape elastic factor (elastic constant other than Young's modulus, Banerjee 1966*a*) of the load.

Duration of impact and its dependence on the striking velocity of the load

In the present analysis the duration of impact ϕ_0 is defined as the time elapsed from the instant the beam and the load come in contact to the instant the contact terminates. In case of "multiple contact" it is therefore the duration of first contact.

Considering Hertz's theory of impact the duration of impact can be divided into three distinct periods, namely First Hertz, Hooke and Second Hertz, respectively, as has been done by Ghosh (1940) and Banerjee (1966*c*).

During each Hertz period, $P = -k_0 u^{3/2}$ ... (5)

During the Hooke period, $P = -E_2 u$ ... (6)

where k_0 = Hertz constant, and u = compression of the load.

The duration of Hooke period ϕ is the lowest positive root of t other than zero, obtained from equations (4) by solving $P = 0$. The duration of each Hertz period τ_0 as given by Ghosh (1940) and Banerjee (1966c) is $\tau_0 = u_0/V_0$; (7) where, u_0 is the limiting value of u at the end of each Hertz period Thus, the total duration of impact is $\phi_0 = \phi + 2\tau_0 = \phi + 2u_0/V_0$ ... (8)

Energy of the beam

From equation (3 1) the velocity of the load at any instant during impact is

$$V_t = \frac{dy_a}{dt} = 4V_0 \Sigma A_s \cos q_s t \quad \text{... (9)}$$

The energy absorbed by the beam is assumed to be the energy lost by the striking load during impact and is given by $\frac{1}{2}m(V_0^2 - V_f^2)$, where V_f = velocity of the load at the termination of final contact and can be obtained from equation (9) The initial energy of the load is $-\frac{1}{2}mV_0^2$. Thus, the energy absorbed ratio of the beam (i.e , the ratio of the energy absorbed by the beam to the initial energy of the load) μ is given by $\mu = 1-(V_f/V_0)^2$ ... (10)

Impact on the beam at its midspan by a large striking mass

When the mass of the striking load is very large in comparison with the mass of the struck beam (i.e m/M is large), equation (1) indicates that for $s = 2, 3,$ etc., γ_s will tend to assume values $5\pi/2$, $9\pi/2$, etc., approximately, and the values of $\gamma_s/(\coth \gamma_s/2 - \cot \gamma_s/2)$ will tend to infinity. So no other terms except the fundamental ($s = 1$) will be present (since $A_s = 0$ for $s = 2, 3, \ldots$ etc) as given by equation (3.4). γ_1 will be approximately $\pi/2$ and the value of $\gamma_1/(\coth \gamma_1/2 - \cot \gamma_1/2)$ will be approximately 3.

Thus, combining equations (3.4) and (9) the final velocity V_f of the beam and the load after the termination of contact for $\gamma_1 = \pi/2$ is

$$V_f = -\frac{V_0}{1+\frac{1}{2}\cdot\frac{M}{m}} \quad \text{... (11)}$$

Equation (11) can be well compared with the expression of Cox given by

$$V_f = -\frac{V_0}{1+\frac{17}{35}\cdot\frac{M}{m}} \quad \text{... (12)}$$

in which (17/35) M is the reduced mass of the beam.

The deflected shape of the beam carrying a concentrated load at midspan ($a = l/2$) can be written as

$$\delta_1 = \delta_a \left[3\left(\frac{x}{l}\right) - 4\left(\frac{x}{l}\right)^3 \right] \qquad \dots \quad (13.1)$$

$$\delta_2 = \delta_a \left[3\left(\frac{l-x}{l}\right) - 4\left(\frac{l-x}{l}\right)^3 \right] \qquad \dots \quad (13.2)$$

where, δ_1 and δ_2 are the static displacement of the beam at any point $x < l/2$ and $x > l/2$ respectively, and δ is the static displacement at $x = a = l/2$. Using Maclaurin's expansion series equations (3 2) and (3.3) for the fundamental mode of vibration ($s = 1$) may be written as

$$y_1 \approx y_a \left[c_1\left(\frac{x}{l}\right) - c_2\left(\frac{x}{l}\right)^3 \right] \qquad \dots \quad (14.1)$$

$$y_2 \approx y_a \left[c_1\left(\frac{l-x}{l}\right) - c_2\left(\frac{l-x}{l}\right)^3 \right] \qquad \dots \quad (14.2)$$

where y_a is given by equation. (3.1)

$$c_1 = \gamma_1 \left[\frac{\sinh \gamma_1 \,.\, \sin\frac{\gamma_1}{2} - \sin \gamma_1 \ \sinh\frac{\gamma_1}{2}}{\sinh \gamma_1 \,.\, \sin^2\frac{\gamma_1}{2} - \sin \gamma_1 \,.\, \sinh^2\frac{\gamma_1}{2}} \right] \qquad \dots \quad (14.3)$$

$$c_2 = \frac{\gamma_1^3}{6} \left[\frac{\sinh \gamma_1 \,.\, \sin\frac{\gamma_1}{2} + \sin \gamma_1 \ \sinh\frac{\gamma_1}{2}}{\sinh \gamma_1 \,.\, \sin^2\frac{\gamma_1}{2} - \sin \gamma_1 \,.\, \sinh^2\frac{\gamma_1}{2}} \right] \qquad \dots \quad (14.4)$$

For $\gamma_1 \leqslant \pi/2$, c_1 and c_2 are found to be approximately 3 and 4, respectively. Thus, as indicated by equations (13) and (14), the deflection curve of the beam during impact due to a large striking mass has almost the same shape as the static deflection curve. With this assumption Cox had developed an expression for dynamic deflection of a simply supported beam due to a falling load at the centre ($a = l/2$) of the beam which may be written as

$$y_a = \delta_a + \left[\delta_a^2 + \delta_a \times \frac{V_0^2}{g\left(1 + \frac{17}{35} \cdot \frac{M}{m}\right)} \right] \qquad \dots \quad (15.1)$$

where, $\delta_a = \frac{mgl^3}{48E_1I}$, g = acceleration due to gravity. With the help of equation (1) the value of m/M for $\gamma_1 = \pi/2$ is found to be 7.4. ... (15.2)

Thus, the equations (12) and (15) as given by Cox theory can be rightly employed if the ratio of the mass of the load to the mass of the beam (*i.e.* m/M) equals or exceeds 7.4. Compared to this the author's analysis is perfectly general and equations (3) and (9) can be utilized for all *mass ratios*.

Experiments

Particulars of beams and hammers used in experiments are given in table 1 and 2 respectively

Table 1. Particulars of beams.

Beam	Material	Length (cm)	Diameter (cm)	Weight (gm)	E_1 (kg/cm²)
B_1	Mild steel	90	1.27	869	$2\ 07 \times 10^6$
B_3	Brass	90	1.27	953	0.95×10^6

Table 2. Particulars of hammers (spherical)

Hammer	Material	Weight (gm)	Radius at contact surface (cm)
H_1	Mild steel	908 0	2.95
H_2	Mild steel	294.4	2 00
H_3	Mild steel	268.5	1.98
H_4	Mild steel	213.6	1 91
H_5	Mild steel	106.0	1.47
H_6	Brass	294 4	2.00

The experimental results of this investigation presented in tables 4 and 5 and figures 2 and 3 are entirely based on the photographic method as has been used by Banerjee (1964). The set up and procedure for obtaining photographs are exactly the same as employed by Das & Mishra (1968). A camera box with a narrow vertical slit in its front face and a tuning fork (100 cycles per sec) are placed parallel to the beam with the slit exactly behind the struck point The hammer is allowed to drop from the desired height above the beam. Light from a carbon arc lamp is focussed to the slit of the camera box to obtain photographs of the motion of the beam and hammer and also of the pointer of the vibrating tuning fork on the running photographic paper pinned on a photo carrier (sliding inside the camera box). A few of the photographs so obtained are shown in figure. 1 (Plate 2) and their details in table 3.

TABLE 3. Details of photographs

Beam-hammer	B_1–H_2	B_1–H_4	B_1–H_1	B_1–H_6	B_1–H_3	B_2–H_2
Striking velocity cm/sec	100	200	200	200	200	200
Reference to figure 1 (Plate 2)	A	B	C	D	E	F

For finding the energy absorbed by the beam during impact due to a particular hammer the experimental arrangement is similar to that used by Banerjee (1966*b*). The velocity V_0 before impact and the velocity V_f after impact are calculated, respectively, from the measured drops and the corresponding rebound heights of the hammer The experimental values of the energy-absorbed ratio μ of the beam (table 6) are obtained from equation (10).

RESULTS

For a particular beam-hammer combination equation (1) is solved graphically and then using the method of successive approximation closer results for the values of γ_s ($s = 1, 2, 3$, etc.) are obtained Accordingly the results of the present theory are given in tables 5 and 6 and figures. 2 to 4.

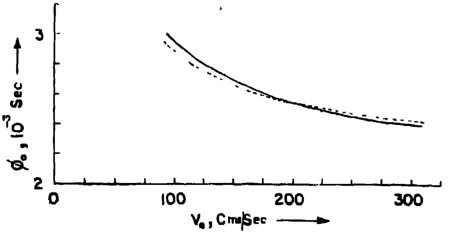

Figure 2. Theoretical (equation 8]. - - - Experimental (from Plate 2. Variation of the duration of impact (ϕ_0) with the striking velocity V_0 for the beam B_1 struck at midspan by hammer H_2.

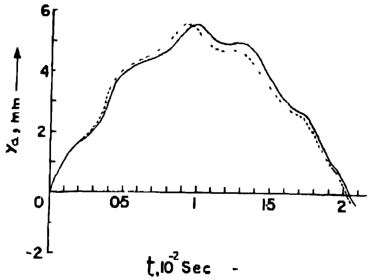

Figure 3 | Theoretical (equation 3 1) --- Experimental (figure 1B Plate 2). Time *vs.* displacement curves of the struck point (midspan). Beam B_1, hammer H_2, $V_0=200$ cms/Sec.

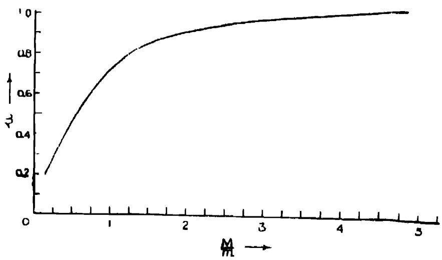

Figure 4. Theoretical curve (equation 10) of the energy-absorbed ratio M, *vs*, 'mass-ratio' M/m for the beam B_1 struck at midspan

TABLE 4. Experimental values of the duration of impact ϕ_0 and maximum displacement (y_a max) at the struck point (midspan) of the beam for different striking velocities V_0

V_0 (cm/sec)		100	150	200	250	300
ϕ_0 (10^{-3} sec)	Beam B_1 Hammer H_2	2 89	2.66	2 55	2.48	2.43
	Beam B_1 Hammer H_b	2 98	2.74	2.62	2.55	2.50
y_a max (mm)	Beam B_1 Hammer H_2	2.762	4 182	5 540	6 934	8.33[illegible]

TABLE 5. Comparison of theoretical and experimental results for various beam-hammer combinations. The beam is struck at midspan with $V_0 = 200$ cm/sec

Beam-hammer		B_1–H_1	B_1–H_2
Maximum displacement at struck point y_a max, mm	Present theory equation (3.1)	12 23	5 492
	Cox Theory equation (15 1)	12 55	5.487
	Experiment	12.27	5 540
One-half of the period of vibration τ, sec	Present theory equation (3.1)	0 0280	0.0204
	Experiment	0.0273	0.0202

TABLE 6 Theoretical and experimental values of the energy-absorbed ratio (μ) of the beam B_1 struck at midspan.

Hammer	Theoretical (Equation 10)	Experimental
H_1	0 7219	0.7396
H_2	0 9581	0 9614

DISCUSSION OF RESULTS

1. For a particular beam-hammer combination the duration of impact diminishes in magnitude as the striking velocity of the hammer increases (table 4 and figure 2)

2 For hammers of different materials having equal weight and equal radius at contact surface striking a particular beam with equal striking velocity the duration of impact is different (table 4). These small variations are due to the difference in the elastic properties of the striking hammers.

3 For the beam struck by a particular hammer with different striking velocities the ratio between the amplitudes of experimental time-displacement curves (and thus the maximum displacemnt) of the beam and the corresponding striking velocity of the hammer is nearly the same (table 4). This is in full agreement with the theory given by equation (3.1).

4. The general shape of vibration curves of a particular beam struck by hammers of equal mass but of different elasticity is almost alike (figures 1B and 1D Plate 2).

5 The vibration patterns of beams of different materials (hence different masses) having same length and cross-section are almost similar, the ratio of the mass of the beam to the mass of the hammer being equal in each case (figures 1E and 1F Plate 2).

6. For a particular beam, as the mass of the hammer decreases, the ratio of the loss of energy to the initial energy of the hammer, *i.e.* the energy-absorbed ratio of the beam tends to assume a constant maximum value (figure 4).

7 For a simply supported beam struck at midspan, the Cox theory can be rightly applied when the ratio of the mass of the load to the mass of the beam equals or exceeds 7.4.

Conclusion

Outside the duration of impact the motion of the beam given by displacement equation (3.1) will be affected due to the detachment of the load and its subsequent contacts with the beam (*i.e.* due to "multiple contact"). Thus the results of the present theory given in figures 3 and 4 and tables 5 and 6 are to be slightly modified. This aspect is at present under the study of the author and will be reported in a subsequent paper.

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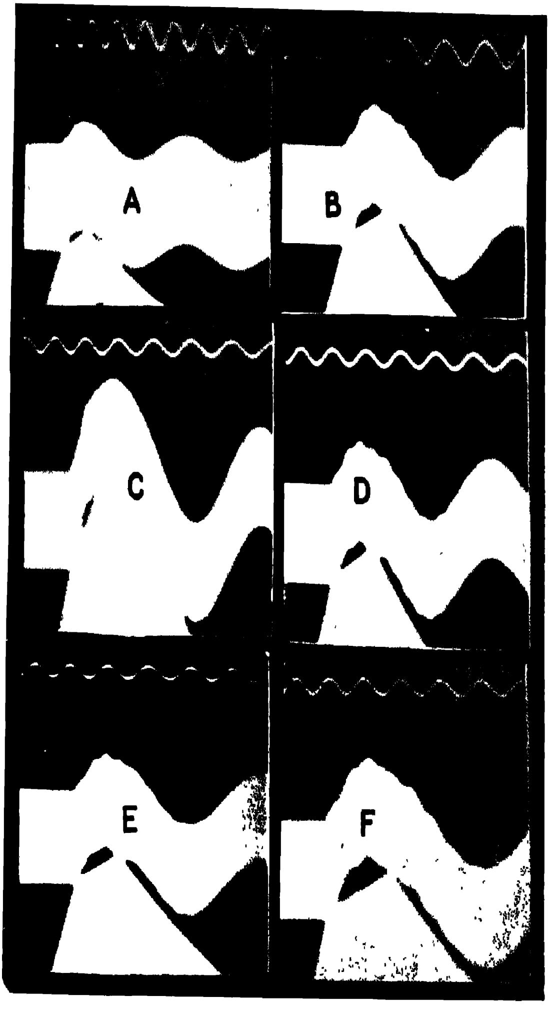

Figure 1. Photographs at the struck point (mid span) of the beams.